# THE COMMUTATION RELATIONS ON THE COVARIANT DERIVATIVE

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Abstract. The algebraic statement which implies the commutation relations on the covariant derivative is proved.

## 1. Introduction

Let $X$ be a smooth manifold. Let $\mathcal{V} = \mathcal{V}(X)$ be a Lie algebra of smooth vector fields and let $\mathcal{D}(X)$ be an algebra of smooth differential operators on $X$. Because of the inclusion $\mathcal{V} \subset \mathcal{D}(X)$ we have the homomorphism

$$\tau : T(\mathcal{V}) \to \mathcal{D}(X).$$

Let

$$p : T(\mathcal{V}) \to U(\mathcal{V})$$

be a natural homomorphism to the universal enveloping algebra of $\mathcal{V}$. Obviously

$$\tau : v \otimes w - w \otimes v - [v, w] \mapsto 0,$$

so the map

$$\tau \circ p^{-1} : U(\mathcal{V}) \to \mathcal{D}(X)$$

is well-defined.

We assume the manifold $X$ to be provided with a smooth linear connection. Then there exists another map $\mu : T(\mathcal{V}) \to \mathcal{D}(X)$, which is not a homomorphism:

$$\mu : T(\mathcal{V}) \to \mathcal{D}(X),\ \mu : v_1 \otimes \cdots \otimes v_n \mapsto v_1 \otimes \cdots \otimes v_n \cdot \nabla^n. \tag{1}$$

The operator at the right hand side of (1) maps the function $f \in C^\infty(X)$ to the convolution of the contravariant tensor $v_1 \otimes \cdots \otimes v_n$ with the covariant tensor $\nabla^n f$ (for example, in the index notation $u \otimes v \otimes w \cdot \nabla^3 = u^i v^j w^k \nabla_i \nabla_j \nabla_k$ ).

The main result of this paper is the explicit expression of the element $R = R(v_1, \ldots, v_n) \in T(\mathcal{V})$ which obeys the equality

$$\mu(v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n - R) = 0. \tag{2}$$

The degree of $R$ is less than $n$. Besides, it may be expressed in terms of the torsion tensor and the curvature tensor of the connection (it is clear that $R$ is not uniquely determined by (2)). Then, the equality (2) means the commutation relations on the covariant derivative. This relations turn out to be fairly complicated.

The idea is to consider the map $K : T(\mathcal{V}) \to T(\mathcal{V})$ which connects $\tau$ and $\mu$: $\mu = \tau \circ K$. For example, an easy computation shows that

$$v \otimes w \cdot \nabla^2 = \nabla_v \nabla_w - \nabla_{\nabla_v w},$$

hence $K(v \otimes w) = v \otimes w - \nabla_v w$.

The map $K$ is invertible but $p \circ K$ has a substantial kernel $\ker(p \circ K) \subset \ker(\mu)$. Moreover, the map

$$p \circ K : T(\mathfrak{g}) \to U(\mathfrak{g})$$

may be defined for any "framed Lie algebra" $\mathfrak{g}$, i.e. a Lie algebra endowed with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. This allows us to solve the problem with algebraic tools.

## 2. The main definitions

We assume all the algebras below to be over some field $\Bbbk$ of zero characteristic. In the geometric case $\Bbbk = \mathbb{R}$.

**Definition** *A framed Lie algebra is a pair of a Lie algebra $\mathfrak{g}$ and a bilinear map*

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

.

Let the sign $\diamond$ denotes this map:

$$(x, y) \mapsto x \diamond y \in \mathfrak{g}.$$

The Lie algebra of vector fields $\mathcal{V}(X)$ on the manifold $X$ with the connection is a framed Lie algebra over $\mathbb{R}$ with $v \diamond w = \nabla_v w$, where $\nabla_v$ is the covariant derivative along $v$.

**Definition** *Let $\mathfrak{g}$ be a framed Lie algebra and $x \in \mathfrak{g}$. The map*

$$\nabla_x : T(\mathfrak{g}) \to T(\mathfrak{g})$$

*is a derivation of the tensor algebra which satisfies the condition*

$$\nabla_x : y \mapsto x \diamond y,\ y \in \mathfrak{g}.$$

For example,

$$\nabla_x : y \otimes z \otimes w \mapsto (x \diamond y) \otimes z \otimes w + y \otimes (x \diamond z) \otimes w + y \otimes z \otimes (x \diamond w).$$

By definition, $\nabla_x 1 = 0$. The author chose the sign $\nabla$ because of the connection between this derivation and the covariant derivative in the case $\mathfrak{g} = \mathcal{V}(X)$.

**Definition** *Let $\mathfrak{g}$ be a framed Lie algebra. The map $K : T(\mathfrak{g}) \to T(\mathfrak{g})$ is such a linear map that $K(1) = 1$ and*

$$K(x \otimes u + \nabla_x u) = x \otimes K(u),\ x \in \mathfrak{g}, u \in T(\mathfrak{g}).$$

It is easy to see that $K$ is uniquely determined by these conditions. If $L_x$ denotes the left tensor multiplication by $x \in \mathfrak{g}$ then the latter relation may be written in the form

$$K L_x = L_x K - K \nabla_x. \tag{3}$$

This recurrence formula enables us to compute this map at any degree:

$$K(x) = K(x \otimes 1) = x,\ K(x \otimes y) = x \otimes y - x \diamond y,$$

$$K(x \otimes y \otimes z) = x \otimes K(y \otimes z) - K((x \diamond y) \otimes z) - K(y \otimes (x \diamond z)), \text{etc.}$$

If $\mathfrak{g} = \mathcal{V}(X)$ and $\tau, \mu$ are as above then

$$\mu = \tau \circ K.$$

It has been proved in [1, Proposition 1]. Actually this relation is a consequence of the well-known formula

$$\nabla_v (U \cdot V) = (\nabla_v U) \cdot V + U \cdot (\nabla_v V),\ v \in \mathcal{V},$$

where $U$ is a contravariant tensor field, $V$ is a covariant tensor field and the dot means the convolution.

We'll use the notation

$$t(x,y) = x \diamond y - y \diamond x - [x,y],$$
$$r(x,y) : z \mapsto x \diamond (y \diamond z) - y \diamond (x \diamond z) - [x,y] \diamond z.$$

If $\mathfrak{g}$ is a framed Lie algebra and $x, y \in \mathfrak{g}$ then $t(x,y) \in \mathfrak{g}$ and $r(x,y) : \mathfrak{g} \to \mathfrak{g}$. In the geometric case

$$t(v,w) = T(v,w),\ r(v,w)u = R(v,w)u,\ v,w,u \in \mathcal{V}(X),$$

where $T$ is the torsion tensor and $R$ is the curvature tensor.

For any space $\mathfrak{g}$ let $\Lambda(\mathfrak{g})$ be a linear subspace of $T^2(\mathfrak{g})$ generated by the elements $x \otimes y - y \otimes x$, $x, y \in \mathfrak{g}$ and $J(\mathfrak{g}) = T(\mathfrak{g})\Lambda(\mathfrak{g})$ be a left ideal in the tensor algebra.

**Definition** *Let $\mathfrak{g}$ be a framed Lie algebra. The linear map*

$$t : J(\mathfrak{g}) \to \mathfrak{g}$$

*is defined by*

$$t(x \otimes y - y \otimes x) = t(x,y) = x \diamond y - y \diamond x - [x,y]$$

*and*

$$t(x \otimes u + \nabla_x u) = x \diamond t(u),\ x \in \mathfrak{g}, u \in J(\mathfrak{g}). \tag{4}$$

The equality (4) may be written in the form

$$tL_x = [\nabla_x, t],\ x \in \mathfrak{g}.$$

For example,

$$t(z \otimes x \otimes y - z \otimes y \otimes x) = z \diamond t(x,y) - t(z \diamond x, y) - t(x, z \diamond y).$$

In the notation of [1],

$$t(z_1 \otimes \cdots \otimes z_n \otimes (x \otimes y - y \otimes x)) = t_{n+2}(z_1, \ldots, z_n, x, y).$$

**Definition** *Let $\mathfrak{g}$ be a framed Lie algebra. The linear map*

$$r : J(\mathfrak{g}) \to \mathrm{End}(\mathfrak{g})$$

*is defined by*

$$r(x \otimes y - y \otimes x) = r(x,y),\ x, y \in \mathfrak{g},$$
$$r(x \otimes u + \nabla_x u) = \nabla_x r(u) - r(u)\nabla_x,\ x \in \mathfrak{g}, u \in J(\mathfrak{g}).$$

In the notation of [1],

$$r(z_1 \otimes \cdots \otimes z_n \otimes (x \otimes y - y \otimes x))w = r_{n+3}(z_1, \ldots, z_n, x, y, w).$$

In fact, for $\mathfrak{g} = \mathcal{V}(X)$ the polynomials $t_n$ and $r_n$ are the (high order) covariant derivatives of of the torsion tensor and the curvature tensor [1, Proposition 5]:

$$t_3(s,v,w) = (\nabla_s T)(v,w),\ r_4(s,v,w,u) = (\nabla_s R)(v,w)u, \text{etc.}$$

This is the cause of their importance.

The map $r(u) : \mathfrak{g} \to \mathfrak{g}$ may be extended to $T(\mathfrak{g})$ by the requirement

$$r(u) \in \mathrm{Der}_{\Bbbk}(T(\mathfrak{g})),\ u \in J(\mathfrak{g});$$
$$r(u) : a \otimes b \mapsto (r(u)a) \otimes b + a \otimes r(u)b, \text{etc.}$$

In this case the equality

$$r(x \otimes u + \nabla_x u) = [\nabla_x, r(u)] \tag{5}$$

remains true.

## 3. The commutation relations

We'll need the coproduct in the tensor algebra:

$$\triangle : T(\mathfrak{g}) \to T(\mathfrak{g}) \hat{\otimes} T(\mathfrak{g})$$

(we use the hat to distinguish the product between two tensor algebras from the common product). This is a well-known homomorphism of associative algebras defined by

$$\triangle : x \mapsto 1 \hat{\otimes} x + x \hat{\otimes} 1,\ x \in \mathfrak{g}.$$

Actually $T(\mathfrak{g})$ is a cocommutative Hopf algebra, but we make no use of antipode. In the Sweedler notation

$$\triangle(u) = u_{(1)} \hat{\otimes} u_{(2)},$$

$$\triangle(z \otimes u) = u_{(1)} \hat{\otimes} z \otimes u_{(2)} + z \otimes u_{(1)} \hat{\otimes} u_{(2)},\ z \in \mathfrak{g}. \tag{6}$$

**Definition** *The linear map $\rho : J(\mathfrak{g}) \to \mathrm{End}(T(\mathfrak{g}))$ is defined by*

$$\rho(u \otimes \omega) : v \mapsto u_{(1)} \otimes (t(u_{(2)} \otimes \omega) \otimes v + r(u_{(2)} \otimes \omega)v),$$

*where $u \in T(\mathfrak{g})$ and $\omega \in \Lambda(\mathfrak{g})$.*

For example,

$$\rho(\omega) : v \mapsto t(\omega) \otimes v + r(\omega)v,$$

$$\rho(z \otimes \omega) : v \mapsto z \otimes (t(\omega) \otimes v + r(\omega)v) + t(z \otimes \omega) \otimes v + r(z \otimes \omega)v,\ z \in \mathfrak{g}.$$

It is easy to see that

$$\deg \rho(u \otimes \omega)v < \deg u \otimes \omega \otimes v.$$

Now we are ready to formulate our main result.

**Theorem** *Let $\mathfrak{g}$ be a framed Lie algebra, $\omega \in \Lambda(\mathfrak{g})$, and $u, v \in T(\mathfrak{g})$. Let $p, K, \rho$ are defined as above. Then*

$$p \circ K(u \otimes \omega \otimes v + \rho(u \otimes \omega)v) = 0. \tag{7}$$

If $\mathfrak{g} = \mathcal{V}(X)$, then from (7) we have

$$\mu(U \otimes (v \otimes w - w \otimes v) \otimes V + \rho(U \otimes (v \otimes w - w \otimes v))V) = 0,$$

where $U, V \in T(\mathcal{V})$ and $v, w \in \mathcal{V}$, so this equality means a commutation relations on the covariant derivative (2) with

$$R = \rho(v_1 \otimes \cdots \otimes v_{i-1} \otimes (v_i \otimes v_{i+1} - v_{i+1} \otimes v_i))v_{i+2} \otimes \cdots \otimes v_n. \tag{8}$$

The author's aim was to express $R$ in terms of $t_n$ and $r_n$. This restriction is of large importance. As a matter of fact the domain of $\mu$ is the space of smooth contravariant tensor fields rather then $T(\mathcal{V})$:

$$T(\mathcal{V}) \to C^\infty(\tau_0^* X) \xrightarrow{\mu} \mathcal{D}(X).$$

Hence $R(v_1, \ldots, v_n)$ should be a polylinear function of $v_i \in \mathcal{V}(X)$ not over $\mathbb{R}$ only but over $C^\infty(X)$:

$$v_1 \underset{C^\infty(X)}{\otimes} v_2 \underset{C^\infty(X)}{\otimes} \cdots \underset{C^\infty(X)}{\otimes} v_n \mapsto R(v_1, \ldots, v_n),$$

otherwise the equality (2) becames meaningless. The term $R$ defined by (8) obeys this requirement because in the geometric case $t_n$ and $r_n$ are tensor invariants of the connection.

In the simplest case $u = 1, \omega = x \otimes y - y \otimes x, v \in \mathfrak{g}$ (7) may be written in the form

$$x \otimes y \otimes v - y \otimes x \otimes v + t(x, y) \otimes v + r(x, y)v \in \ker(p \circ K),$$

which may be computed directly. Applying the map $\mu$, we obtain

$$v_1 \otimes v_2 \otimes w \cdot \nabla^3 - v_2 \otimes v_1 \otimes w \cdot \nabla^3 + T(v_1, v_2) \otimes w \cdot \nabla^2 + R(v_1, v_2)w \cdot \nabla = 0,$$

or in the index notation

$$\nabla_i \nabla_j \nabla_k - \nabla_j \nabla_i \nabla_k + T^l_{ij} \nabla_l \nabla_k + R^l_{ijk} \nabla_l = 0.$$

## 4. Another definitions

We assume $\mathfrak{g}$ to be a framed Lie algebra below.

**Definition** *The linear map*

$$e : J(\mathfrak{g}) \to T(\mathfrak{g})$$

*is defined by*

$$e(x \otimes y - y \otimes x) = x \otimes y - y \otimes x - [x, y],$$
$$e(x \otimes u + \nabla_x u) = x \otimes e(u) - e(u) \otimes x,\ x \in \mathfrak{g}, u \in J(\mathfrak{g}). \tag{9}$$

**Lemma 1** *Let $e : J(\mathfrak{g}) \to T(\mathfrak{g})$ and $p : T(\mathfrak{g}) \to U(\mathfrak{g})$ are defined as above. Then $p \circ e = 0$.*

Obviously $p \circ e(x \otimes y - y \otimes x) = 0$. For any $u \in J(\mathfrak{g})$ and $x \in \mathfrak{g}$ we have

$$p \circ e(x \otimes u) = [x, p \circ e(u)] - p \circ e(\nabla_x u).$$

Hence $p \circ e = 0$ follows by induction on the $\deg(u)$.

There is a relation between $e$ and $r$. While it has nothing to do with the theorem, it has some meaning on it's own.

**Proposition** *Let $\nabla : T(\mathfrak{g}) \to \mathrm{End}(T(\mathfrak{g}))$ be a homomorphism defined by the condition $\nabla : x \to \nabla_x$, $x \in \mathfrak{g}$. Then $r = \nabla \circ e$.*

It is easy to see that

$$r(x \otimes y - y \otimes x) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}. \tag{10}$$

Actually, the right hand side of (10) is a derivation of $T(\mathfrak{g})$, which maps $z \in \mathfrak{g}$ to $r(x, y)z$. Hence,

$$r(\omega) = \nabla \circ e(\omega),\ \omega \in \Lambda(\mathfrak{g}).$$

By (9) we have

$$\nabla(e(x \otimes u + \nabla_x u)) = \nabla(x \otimes e(u) - e(u) \otimes x) = [\nabla_x, \nabla \circ e(u)].$$

Hence the map

$$\nabla \circ e : J(\mathfrak{g}) \to \mathrm{End}(T(\mathfrak{g}))$$

satisfies the same equality (5) as the map $r$. Then $r = \nabla \circ e$ by induction.

**Definition** *The linear map $\kappa : J(\mathfrak{g}) \to \mathrm{End}(T(\mathfrak{g}))$ is defined by*

$$\kappa(u \otimes \omega) : v \mapsto e(u_{(1)} \otimes \omega) \otimes K(u_{(2)} \otimes v),$$

*where $u \in T(\mathfrak{g})$ and $\omega \in \Lambda(\mathfrak{g})$.*

For example,

$$\kappa(x \otimes y - y \otimes x) : v \mapsto (x \otimes y - y \otimes x - [x,y]) \otimes K(v).$$

## 5. Proof of the theorem

**Lemma 2** *Let $\omega \in \Lambda(\mathfrak{g})$ and $v \in T(\mathfrak{g})$. Then*

$$K(\omega \otimes v + \rho(\omega)v) = \kappa(\omega)v. \qquad (11).$$

The equality (11) is equivalent to

$$K[L_x, L_y] + KL_{t(x,y)} + Kr(x,y) = ([L_x, L_y] - L_{[x,y]})K.$$

The latter follows from (3), (5) and

$$L_{t(x,y)} = [\nabla_x, L_y] - [\nabla_y, L_x] - L_{[x,y]}.$$

**Lemma 3** *Let $z \in \mathfrak{g}$, $Q \in J(\mathfrak{g})$. Then*

$$\rho(z \otimes Q + \nabla_z Q) = L_z\rho(Q) + [\nabla_z, \rho(Q)]. \qquad (12)$$

We define

$$\eta : J(\mathfrak{g}) \to \mathrm{End}(T(\mathfrak{g})),\ \lambda : \mathfrak{g} \to \mathrm{End}(T(\mathfrak{g}))$$

by

$$\eta(Q) = L_{t(Q)} + r(Q),\ Q \in J(\mathfrak{g}),$$
$$\lambda_z = L_z + \nabla_z,\ z \in \mathfrak{g}.$$

Then

$$L_{\lambda_z u} = \lambda_z L_u - L_u \nabla_z, \qquad (13)$$
$$\triangle(\lambda_z u) = (\lambda_z u_{(1)})\hat{\otimes} u_{(2)} + u_{(1)} \hat{\otimes} \lambda_z u_{(2)}, \qquad (14)$$

where $z \in \mathfrak{g}$, $u \in T(\mathfrak{g})$. By (4) and (5) we have

$$\eta(z \otimes Q + \nabla_z Q) = [\nabla_z, \eta(Q)]. \qquad (15)$$

We may assume

$$Q = u \otimes \omega,\ u \in T(\mathfrak{g}),\ \omega \in \Lambda(\mathfrak{g}). \qquad (16)$$

Then

$$\lambda_z Q = (\lambda_z u) \otimes \omega + u \otimes \nabla_z \omega, \qquad (17)$$
$$\rho(Q) = \rho(u \otimes \omega) = L_{u_{(1)}} \eta(u_{(2)} \otimes \omega). \qquad (18)$$

By (14), (17) and (18) we have

$$\rho(\lambda_z Q) = L_{\lambda_z u_{(1)}} \eta(u_{(2)} \otimes \omega) + L_{u_{(1)}} \eta(\lambda_z(u_{(2)} \otimes \omega)).$$

By (13) and (15) we have

$$\rho(\lambda_z Q) = \lambda_z \rho(Q) - L_{u_{(1)}} \nabla_z \eta(u_{(2)} \otimes \omega) + L_{u_{(1)}} [\nabla_z, \eta(u_{(2)} \otimes \omega)] = \lambda_z \rho(Q) - \rho(Q)\nabla_z.$$

Hence (12) follows.

**Lemma 4** *Let $z \in \mathfrak{g}$, $Q \in J(\mathfrak{g})$. Then*

$$\kappa(z \otimes Q + \nabla_z Q) = L_z \kappa(Q) - \kappa(Q)\nabla_z. \qquad (19)$$

Assuming (16) we have

$$\kappa(Q) = L_{e(u_{(1)} \otimes \omega)} K L_{u_{(2)}}.$$

By (17) and (14) we have

$$\kappa(\lambda_z Q) = \kappa((\lambda_z u) \otimes \omega + u \otimes \nabla_z \omega) = L_{e(\lambda_z(u_{(1)} \otimes \omega))} K L_{u_{(2)}} + L_{e(u_{(1)} \otimes \omega)} K L_{\lambda_z u_{(2)}}.$$

Using

$$K\lambda_z = L_z K,\ e(\lambda_z u) = z \otimes e(u) - e(u) \otimes z$$

and (13) we deduce that

$$\kappa(\lambda_z Q) = (L_{e(\lambda_z(u_{(1)} \otimes \omega))} + L_{e(u_{(1)} \otimes \omega)} L_z) K L_{u_{(2)}} - \kappa(Q)\nabla_z = L_z \kappa(Q) - \kappa(Q)\nabla_z.$$

**Lemma 5** *Let $u, v \in T(\mathfrak{g})$ and $\omega \in \Lambda(\mathfrak{g})$. Then*

$$K(u \otimes \omega \otimes v + \rho(u \otimes \omega)v) = \kappa(u \otimes \omega)v. \tag{20}$$

Write

$$Z(Q) = KL_Q + K\rho(Q) - \kappa(Q),\ Q \in J(\mathfrak{g}).$$

Then (20) means $Z(Q) \equiv 0$. By Lemma 2, it holds for $Q \in \Lambda(\mathfrak{g})$. By (12), (19) and (13) we obtain

$$Z(\lambda_z Q) = K\lambda_z L_Q - KL_Q\nabla_z + K\lambda_z\rho(Q) - K\rho(Q)\nabla_z - L_z\kappa(Q) + \kappa(Q)\nabla_z =$$
$$= L_z Z(Q) - Z(Q)\nabla_z.$$

Hence (20) follows by induction on $\deg(Q)$.

Applying $p$ to (20) by Lemma 1 we have (7). The theorem is proved.

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